# Solutions of Restricted Three-Body Problem Represented by Means of Two-Fixed-Center Problem

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THE restricted three-body problem asks for the description of the general motion of a particle $P$ of negligible mass moving subject to the Newtonian attraction from two other bodies $P_1$ and $P_2$ with positive masses which rotate in circles about their center of gravity. The two-dimensional case only will be considered, where $P$ moves in the plane of $P_1$, $P_2$, and the equations of motion will be established first in a convenient form.

Let $(x, y)$ be the coordinates of $P$ in a fixed Cartesian coordinate system of the two-dimensional Euclidean space with its origin at the c.m. of $P_1$, $P_2$. The units of time, mass, and length are chosen such that the gravitational constant is 1, the masses of $P_1$ and $P_2$ are $1 - \mu$ and $\mu$ with $0 < \mu < 1$, and the distance between $P_1$ and $P_2$ equals 1. The position vectors of $P$, $P_1$, $P_2$ can be characterized by complex numbers and then are determined as $z = x + iy$, $-\mu e^{it}$, $(1 - \mu)e^{it}$. The latter two actually describe the circular motion of $P_1$, $P_2$ under their Newtonian attraction. Then (with dots denoting derivatives by time $t$)

$$\ddot{z} = -(1 - \mu)\frac{z + \mu e^{it}}{|z + \mu e^{it}|^3} - \mu\frac{z + (\mu - 1)e^{it}}{|z + (\mu - 1)e^{it}|^3} \qquad [1]$$

is the equation of motion of $P$. The coordinate transformation $z = x + iy = e^{it}(x_1 + ix_2)$ to a rotating Cartesian coordinate system, in which $P$, $P_1$, $P_2$ have the coordinates $(x_1, x_2)$, $(-\mu, 0)$, $(1 - \mu, 0)$ leads to the differential equations of the plane restricted three-body problem:

$$\begin{aligned} \ddot{x}_1 &= 2\dot{x}_2 + x_1 + U_{x_1} \\ \ddot{x}_2 &= -2\dot{x}_1 + x_2 + U_{x_2} \end{aligned} \qquad [2]$$

with

$$\begin{aligned} U &= (1 - \mu)/r_1 + \mu/r_2 \\ r_1 &= [(x_1 + \mu)^2 + x_2^2]^{1/2} \\ r_2 &= [(x_1 + \mu - 1)^2 + x_2^2]^{1/2} \end{aligned} \qquad [3]$$

where variables as subscripts denote the corresponding partial derivatives, here and in the following. With the Hamiltonian function

$$H = H(x, y) = \tfrac{1}{2}(y_1^2 + y_2^2) + x_2y_1 - x_1y_2 - U \qquad [4]$$

one can rewrite [2] in canonical form

$$\dot{x}_k = H_{y_k} \qquad \dot{y}_k = -H_{x_k} \qquad (k = 1, 2) \qquad [5]$$

Besides $H$, consider the simpler Hamiltonian function

$$E = E(x, y) = \tfrac{1}{2}(y_1^2 + y_2^2) - U \qquad [6]$$

with $U$ from [3]. Replacing $H$ with $E$ in [5] gives (in the second coordinate system, in which $P_1$ and $P_2$ remain fixed) equations of motion for $P$, where $P$ is subjected to the Newtonian attraction from $P_1$ and $P_2$ alone, and not also to centrifugal and coriolis forces as in [5] using [4]. Thus one gets the so-called Euler problem of two fixed centers, which is solvable by a transformation to elliptic coordinates and a new time variable in elliptic functions (Euler, 1765); see Ref. 1. This fact becomes important in the analytic representation of the solutions of the restricted three-body problem by means of the solutions of Euler's problem to be given in Eqs. [17–21].

Matrix notation is introduced next. Let $z$ be the (matrix) column with elements $x_1$, $x_2$, $y_1$, $y_2$; (against the foregoing notation) similarly let $w$ be the column of $u_1$, $u_2$, $v_1$, $v_2$; $H_z'$ be the column of the partial derivatives $H_{x_1}$, $H_{x_2}$, $H_{y_1}$, $H_{y_2}$; and $H = H(x, y) = H(z)$; similarly, $E_w'$ if $E = E(w)$. Finally, define a $4 \times 4$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad [7]$$

where $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then [5] can be written as $\dot{z} = JH_z'$; similarly, $\dot{w} = JE_w'$ is the differential equation of the Euler problem, if $E = E(w)$ is given by [6] with $w$ instead of $z$.

Let $\zeta$ be the column with elements $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$, which are again real variables but subject to the conditions $(-\mu,0) \neq (\xi_1, \xi_2) \neq (1 - \mu,0)$. Then, according to the existence theorems, there is determined a unique function $w(t,\zeta)$ as solution of Euler's problem with the initial values $\zeta$, satisfying (for sufficiently small $|t|$ depending on $\zeta$)

$$w_t(t,\zeta) = JE_w'[w(t,\zeta)] \qquad w(0,\zeta) = \zeta \qquad [8]$$

For fixed initial values $\zeta^*$, the solution $w(t, \zeta^*)$ can be continued analytically along the real $t$ axis as long as $(-\mu, 0) \neq [u_1(t,\zeta^*),u_2(t,\zeta^*)] \neq (1 - \mu,0)$ for $t \geqq 0$, which means until a collision occurs with $P_1$ or $P_2$. If $w(t,\zeta^*)$ is holomorphic for $0 \leqq t \leqq t^*$, say, then $E$ is holomorphic in the variables $u_1$, $u_2$, $v_1$, $v_2$ on the set given by $w = w(t,\zeta^*)$ for $0 \leqq t \leqq t^*$, and as a consequence $w(t,\zeta)$ is a holomorphic function of $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$, and $t$ for $\zeta$ in some four-dimensional complex neighborhood of $\zeta^*$ and $0 \leqq t \leqq t^*$ (see Ref. 2). Thus $w(t,\zeta)$ especially has continuous partial derivatives respective to $t$ or $\zeta$ in some domain of the five-dimensional real $t$, $\xi_1$, $\xi_2$, $\eta_1$, $\eta_2$ space, and one can assume that, with a point $(t^*,\zeta)$, $t^* > 0$, also all points $(t,\zeta)$ for $0 \leqq t \leqq t^*$ belong to this domain, which is denoted by $D$. For fixed real $t$, let $D_t$ denote the set of all four-dimensional points $\zeta$, for which $(t,\zeta)$ belongs to $D$. Then by

$$w(t,\zeta) = z \qquad [9]$$

for $\zeta$ in $D_t$, there is given a mapping of $D_t$ into the four-dimensional real $x_1$, $x_2$, $y_1$, $y_2$ space, with the unique inverse $\zeta = w(-t,z)$, since $E$ does not depend explicitly on $t$ and because of the uniqueness of the solutions of [8]. Furthermore, this mapping determines a canonical transformation between $\zeta$ and $z$; namely, the functional matrix $M$ of the mapping [9] is symplectic (2);[3] that is,

$$M = M(t,\zeta) = w_\zeta(t,\zeta) \qquad [10]$$

satisfies $M'JM = J$ identically for $(t,\zeta)$ in $D$. Here a prime denotes the transposed matrix, and $M$ consists of the (matrix) rows $u_{1_\zeta}$, $u_{2_\zeta}$, $v_{1_\zeta}$, $v_{2_\zeta}$, using the notation introduced earlier. For the proof of this statement take partial derivatives respective $\zeta$ in [8] and obtain

$$M_t = w_{\zeta t} = w_{t\zeta} = JE_{ww}'M$$

where $E_{ww}' = E_{ww}$ is similarly defined as $M$ and is symmetric. Thus, putting $Q = M'JM = Q(t,\zeta)$,

$$Q_t = M_t'JM + M'JM_t = M'E_{ww}J'JM + M'JJE'_{ww}M = 0$$

since $J' = J^{-1} = -J$. Hence $Q$ is independent of $t$, but $M(0,\zeta)$ is the unit matrix by [8] and [10]; thus $Q(t,\zeta) = Q(0,\zeta) = J$ for all $t,\zeta$ for which $M$ is defined.

By the canonical transformation [9], new dependent variables $\zeta$ are introduced now into [5]. To obtain the result

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ing canonical differential equations for $\zeta$ and the corresponding Hamiltonian, let $z(t,z_0)$ be the uniquely determined solution of [5], with initial values $z_0$:

$$z_t(t,z_0) = JH_z'[z(t,z_0)] \qquad z(0,z_0) = z_0 \qquad [11]$$

and define $\zeta(t,z_0) = w[-t,z(t,z_0)]$, so that $\zeta(0,z_0) = z_0$ and $z(t,z_0) = w[t,\zeta(t,z_0)]$.

Then

$$JH_z'[w(t,\zeta)] = w_t(t,\zeta) + w_\zeta(t,\zeta)\zeta_t(t,z_0) = JE_w'[w(t,\zeta)] + M\dot{\zeta}$$

where $\zeta = \zeta(t,z_0)$. Using [4] and [6], put

$$F(w) = H(w) - E(w) = u_2v_1 - u_1v_2$$
$$\tilde{F}(t,\zeta) = F[w(t,\zeta)] \qquad [12]$$

Then with [10] and the foregoing

$$\tilde{F}_\zeta' = M'F_w' = M'J^{-1}M\dot{\zeta} = -J\dot{\zeta},\ \dot{\zeta} = J\tilde{F}_\zeta'$$

or

$$\zeta_t(t, z_0) = J\tilde{F}_\zeta'[t,\zeta(t,z_0)] \qquad \zeta(0,z_0) = z_0 \qquad [13]$$

By [6] and [8], abbreviating $w(t,\zeta)$ by $w$, etc.,

$$u_{1t} = v_1 \qquad v_{1t} = -\frac{1-\mu}{r_1^3}(u_1+\mu) - \frac{\mu}{r_2^3}(u_1+\mu-1)$$

$$u_{2t} = v_2 \qquad v_{2t} = -\frac{1-\mu}{r_1^3}u_2 - \frac{\mu}{r_2^3}u_2$$

Hence by [12], $\tilde{F}(t,\zeta) = u_2u_{1t} - u_1u_{2t}$ is the negative rotational impulse around the origin of a particle at time $t$, which moves along a Euler trajectory determined by [8] coming from the initial values $\zeta$. Now

$$\tilde{F}_t = u_2v_{1t} - u_1v_{2t} = \mu(1-\mu)(r_2^{-3} - r_1^{-3})u_2$$
$$\tilde{F}(0,\zeta) = F(\zeta)$$

hence by [12]

$$\tilde{F}(t,\zeta) = \xi_2\eta_1 - \xi_1\eta_2 + \mu(1-\mu)\int_0^t (r_2^{-3} - r_1^{-3})u_2(s,\zeta)ds \qquad [14]$$

where under the integral $r_k$ stands for $r_k[u_1(s,\zeta),\ u_2(s,\zeta)]$, $(k = 1,2)$ as determined by [3] with $u_k(s,\zeta)$ instead of $x_k$, and by the solution $w(s,\zeta)$ of [8] with $s$ instead of $t$.

Now let $\gamma$ be the column with elements $\alpha_1, \alpha_2, \beta_1, \beta_2$, which are real variables again. The canonical system $\dot{\gamma} = JF_\gamma'$ with $F = F(\gamma) = \alpha_2\beta_1 - \alpha_1\beta_2$ from [12] has the general solution $\gamma = \gamma(t) = R(t)\gamma(0)$, where the $4 \times 4$ matrix $R(t)$ is given by

$$R(t) = \begin{pmatrix} S(t) & 0 \\ 0 & S(t) \end{pmatrix}$$

$$S(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad [15]$$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Defining

$$G(t,\zeta) = \tilde{F}(t,\zeta) - F(\zeta)$$
$$\tilde{G}(t,\gamma) = G[t,R(t)\gamma] \qquad [16]$$

then the canonical transformation $\zeta = R(t)\gamma$ leads from [13] to the canonical system $\dot{\gamma} = J\tilde{G}_\gamma'$, similarly as [9] led from [5] or [11] to [13]. Thus one obtains the representation

$$z(t,z_0) = w[t,\zeta(t,z_0)] = w[t,R(t)\gamma(t,z_0)] \qquad [17]$$

where the function $\gamma$ is uniquely determined by

$$\gamma_t(t,z_0) = J\tilde{G}_\gamma'[t,\gamma(t,z_0)] \qquad \gamma(0,z_0) = z_0 \qquad [18]$$

the latter by [13] and since $R(0)$ is the unit matrix. Here by [12, 14, and 16]

$$\tilde{G}(t,\gamma) = \mu(1-\mu)\int_0^t (r_2^{-3} - r_1^{-3})u_2[s,R(t)\gamma]ds \qquad [19]$$

where $r_1,r_2$ are defined similarly as in [14] but with $R(t)\gamma$ in place of $\zeta$. By the remarks preceding [9], the representation [17] of the general solution of the restricted three-body problem [4] and [5] is valid as long as $\zeta = R(t)\gamma(t,z_0)$ is in $D_t$.

From [18] and [19], $\gamma(t,z_0)$ has a Taylor expansion by powers of $t$ of the form

$$\gamma(t,z_0) = z_0 + \gamma^{(2)}(z_0)t^2 + \gamma^{(3)}(z_0)t^3 + \ldots \qquad [20]$$

with positive radius of convergence. Therefore, the simplified function $w[t,R(t)z_0]$ of $t$ represents an osculating approximation to $z(t,z_0)$ of [11] for $t = 0$ at any $z_0$.

For small $\mu$ (in case of $P_1$, $P_2$ denoting earth, moon one has $\mu \approx \frac{1}{82}$), [17] and [19] show that $\tilde{G}(t,\gamma)$ remains small as long as the two-dimensional points, whose coordinates are the first two elements of the columns $z(t,z_0)$ or $R(t)\gamma(t,z_0)$, do not approach $P_1$ or $P_2$. Then $\gamma(t,z_0)$ stays nearly equal to $z_0$ by [18], since $\tilde{G}(t,\gamma)$ is a holomorphic function of $\gamma$ especially, which implies an estimate of $\tilde{G}_\gamma(t,\gamma)$ in terms of the maximum of $|\tilde{G}(t,\gamma)|$.

Since $\tilde{G}(t,\gamma)$ is a rather complicated function, numerical investigations have been made into the behavior of $\gamma(t,z_0)$ using a high speed digital computer (IBM 7090). By numerical integration of [11] and then essentially of [8], the authors calculated

$$\gamma(t,z_0) = R(-t)w[-t,z(t,z_0)] = z_0 + \delta(t,z_0) \qquad [21]$$

rather than integrating [18]. These calculations, performed for a multitude of trajectories in the earth-moon case, show that in fact $\delta(t,z_0)$ remains practically zero for time ranges $0 \leqq t \leqq t^*$, with $t^*$ roughly from $\frac{1}{2}$ to 1 depending on the trajectories, which started not too near earth and were chosen to achieve a near lunar circumnavigation. Roughly, $t^*$ resulted to be near the time of closest approach to the moon, and $\delta(t,z_0)$ starts to deviate sharply from 0 for $t > t^*$. Also, one can determine empirically an approximate expression for $\delta(t,z_0)$ valid for different $z_0$ and thus represent by [17] and [21] approximately certain families of earth-moon trajectories. Then expressing the solution of [8] in terms of theta functions yields an analytic representation of this approximation to the solutions of [11] over a finite time range.

Finally, it is of particular interest to observe the following "functional equation":

$$z[t,R(s)z_0] \approx R(s)z(t,z_0) \qquad [22]$$

which is approximately valid with surprising accuracy for real $t$ and $s$ with $|t| + |s| \leqq t^*$, if $\gamma(t,z_0) \approx z_0$ for $|t| \leqq t^*$. This was found empirically but can also be understood from the following approximate consideration. By [19], $\tilde{G}(t,\gamma)$ is a holomorphic function of $t$ and $\gamma$ for $R(t)\gamma$ in $D_t$. The assumption $\gamma(s,z_0) = z_0$ for $0 \leqq s \leqq t^*$ implies $\tilde{G}_\gamma(s,z_0) = 0$ by [18]; thus $\tilde{G}_\gamma(s,z)$ is nearly zero and then $\gamma(s,z) \approx z$ for sufficiently small $|z - z_0|$ and real $s$ with $R(s)\gamma(s,z) \approx R(s)z$ in $D_s$, $0 \leqq s \leqq t^*$. Therefore, if $|t|$ is sufficiently small, then

$$\gamma[s,z(t,z_0)] \approx z(t,z_0)$$

for $R(s)z(t,z_0)$ in $D_s$. But this holds also for sufficiently small $|s|$ by [20]; hence it holds under the assumptions made for [22] with suitable $t^*$. Similarly

$$\gamma[t,R(s)z_0] \approx R(s)z_0$$

if $R(t)R(s)z_0$ in $D_t$. This and [17] imply

$$\begin{aligned} w[s,R(s)z(t,z_0)] &\approx w\{s,R(s)\gamma[s,z(t,z_0)]\} = z[s,z(t,z_0)] \\ &= z(t+s,z_0) = w[t+s,\ R(t+s)\gamma(t+s,z_0)] \\ &\approx w[t+s,\ R(t+s)z_0] \end{aligned}$$

Hence, with $R(t+s) = R(t)R(s)$ from [15],

$$R(s)z(t,z_0) \approx w[t,R(t)R(s)z_0] \approx w\{t,R(t)\gamma[t,R(s)z_0]\} = z[t,R(s)z_0]$$

which is [22].

The functional equation can be visualized easily, since by [15], $R(t)z$ is equivalent with $S(t)x$, $S(t)y$ for $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $S(t)x$ represents a rotation of the position vector $x$ around the origin through the angle $-t$. If $z_0$ is not too near $P_1$ or $P_2$, then by [22] one can generate and dynamically describe a family of solutions $z[t,R(s)z_0]$ of the restricted three-body problem [5] which begin near $P_1$ and end near $P_2$ for $t_1 \leqq t \leqq t_2$ $(t_1 < 0 < t_2)$, for instance, using $s$ as parameter. It is not difficult using the foregoing consideration and [21] practically to determine suitable ranges for $s$, $t_1$, $t_2$, if $z(t, z_0)$ is known.

It is finally remarked that [22] is not purely local in nature but describes a property of the trajectories of [2] and thus also of [1] "in the large." This demonstrates at least one use of [17] and [20] and is related to the physical meaning of $\tilde{F}(t,\zeta)$ mentioned before [14]. Further investigation of the properties of this function as given by [12] deserves an effort in view of [9, 11, 13], and of Ref. 1.

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# Linearized Steady Motion of Pluri-Reacting Mixtures[1]

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In Ref. 1 it has been proved that, when the chemical characteristic time is much smaller than the macroscopic characteristic time, the linearized steady flow of a singly reacting mixture is equivalent to that of an inert gas mixture with volume viscosity. Parallel statements for unsteady motion in the so-called acoustical approximation are proved in Ref. 2.

It is the purpose of this note to prove two similar statements for the linearized steady motion of mixture in which $n$ reactions take place; namely: 1) when all pertinent relaxation times are much smaller than the macroscopic characteristic time, the linearized steady flow of a pluri-reacting mixture is equivalent to that of an inert gas mixture with a volume viscosity depending on the equilibrium state of the mixture; 2) when $m$ relaxation times $(1 \leq m < n)$ are much smaller than the macroscopic characteristic time, the linearized steady flow of an $n$ reacting mixture is equivalent to that of a "viscous" mixture in which $(n - m)$ reactions take place, the equivalent kinematic volume viscosity being the same as that for case 1.

To prove these statements, one describes the local thermodynamic state of the mixture in terms of the independent variables $v$, $s$, $\mathbf{A}$, where $v$ and $s$ are specific volume and entropy, respectively, and $\mathbf{A}$ is an $n$ dimensional vector whose $n$ components $A_i$ are the affinities of the $n$ reactions taking place in the mixture.

All needed thermodynamic information is then given by the first-order homogeneous function $\varphi = \varphi(\mathbf{A}, s, v)$ (the "fundamental relation"), where $\varphi = e - \mathbf{A}\cdot\boldsymbol{\xi}$ is a thermodynamic potential defined (see Ref. 3) as the $n$th order Lagrange transform of the specific energy $(e)$ with respect to the $n$ dimensional vector $\boldsymbol{\xi}$ whose $i$th component $\xi_i$ is the progress variable of the $i$th reaction. The conjugate dependent quantities $\boldsymbol{\xi}$, $T$ (temperature), and $p$ (pressure) are functions of the basic set $(\mathbf{A}, s, v)$ and are defined in terms of the partial derivatives of $\varphi$ according to the Gibbs relation:

$$d\varphi = -\boldsymbol{\xi}\cdot d\mathbf{A} - pdv + Tds \qquad [1]$$

In the fluid-dynamic evolution of a system not too far from equilibrium conditions, the local thermodynamic state can be considered as a first-order deviation (subscript 1) from an equilibrium state (subscript 0), identified by the set $(0, v_0, s_0)$. The first-order deviation, when all molecular transport effects are negligible, is identified by the set $(\mathbf{A}_1, v_1, 0)$; the first-order disturbances $\boldsymbol{\xi}_1$, $p_1$, $T_1$ are linear combinations of $(\mathbf{A}_1, v_1, 0)$ through the second-order derivatives of $\varphi$ computed at the "point" $(0, v_0, s_0)$. Introducing the $(n \times n)$ matrix $\boldsymbol{\Phi}$ and the $n$ dimensional vector $\boldsymbol{\varphi}$ defined by

$$\phi_{ij} = -(\partial^2\varphi/\partial A_i\partial A_j)_0 \qquad \varphi_i = (\partial^2\varphi/\partial A_i\partial v) \qquad i, j\text{: } 1, \ldots n \qquad [2]$$

where the subscript (0) indicates values computed at $(0, v_0, s_0)$, it is

$$p_1 = -\boldsymbol{\varphi}\cdot\mathbf{A}_1 - (a_{e0}^2/v_0)(v_1/v_0) \qquad [2a]$$

$$\boldsymbol{\xi}_1 = \boldsymbol{\Phi}\cdot\mathbf{A}_1 - \boldsymbol{\varphi}v_1 \qquad [2b]$$

where $a_{e0}^2 = v_0^2(\partial^2\varphi/\partial v^2)_{\mathbf{A}=0}$ is, by definition, the "equilibrium" speed of sound. The matrix $\boldsymbol{\Phi}$ is positive definite upon the intrinsic stability of the system.

The basic conservation equations for the first-order, steady disturbances introduced by initial and/or boundary conditions into an otherwise constant property equilibrium flow field can then be written, if $\mathbf{V}$ is the velocity vector and $D/Dt = \mathbf{V}_0\cdot\nabla$, as

$$\begin{gathered}(D\mathbf{V}_1/Dt) - (a_{e0}^2/v_0)\nabla v_1 = \nabla(\boldsymbol{\varphi}\cdot\mathbf{A}_1)\\ Dv_1/Dt = v_0\nabla\cdot\mathbf{V}_1 \qquad Ds_1/Dt = 0\\ D\mathbf{A}_1/Dt = -\boldsymbol{\Phi}^{-1}\cdot\mathbf{L}\cdot\mathbf{A}_1 + \boldsymbol{\Phi}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt)\end{gathered} \qquad [3]$$

where $\boldsymbol{\Phi}^{-1}$ is the inverse matrix of $\boldsymbol{\Phi}$, and the last equation follows from the expression $D\boldsymbol{\xi}_1/Dt = -\mathbf{L}\cdot\mathbf{A}_1$ for the substantial time rate of change of $\boldsymbol{\xi}_1$ and from Eq. [2b]. The $(n \times n)$ matrix $\mathbf{L} = \mathbf{L}(v_0, s_0)$ is positive definite (upon the positive character of the entropy production) and symmetric (upon the Onsager's relation).

Eqs. [1–3] make it possible to prove the forementioned statements. Identify first the "pertinent" relaxation times. Diagonalize the matrix $\mathbf{L}^{-1}\cdot\boldsymbol{\Phi}$ (where $\mathbf{L}^{-1}$ is the inverse of the matrix $\mathbf{L}$) through a similarity transformation, i.e., let

$$\mathbf{B}_1 = \mathbf{N}\cdot\mathbf{A}_1 \qquad \mathbf{N}\cdot(\mathbf{L}^{-1}\cdot\boldsymbol{\Phi})\cdot\mathbf{N}^{-1} = \boldsymbol{\Lambda} \qquad (\Lambda_{ij} = \lambda_i\delta_{ij})$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

Performing these substitutions, the last of Eqs. [3] becomes

$$\mathbf{B}_1 = \mathbf{N}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) - \boldsymbol{\Lambda}\cdot(D\mathbf{B}_1/Dt) \qquad [4]$$

which shows that $\boldsymbol{\Lambda}$ has the dimension of time. On the other hand, by known matrix theorems, the $\lambda_i$'s are the $n$ roots of the determinantal equation

$$|\mathbf{L}^{-1}\cdot\boldsymbol{\Phi} - \lambda\mathbf{U}| = 0$$

($\mathbf{U}$ is the unit matrix), and, since both $\boldsymbol{\Phi}$ and $\mathbf{L}^{-1}$ are positive

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